$\boxed{\text{① (a)}}$

$$\underbrace{(2x-1)}_{M}+\underbrace{(3y+7)}_{N}\frac{dy}{dx}=0$$

Let

$$\left.\begin{aligned} M(x,y)&=2x-1\\ N(x,y)&=3y+7\end{aligned}\right\}\ \text{these are continuous everywhere}$$

We have

$$\left.\begin{aligned} \frac{\partial M}{\partial x}&=2, \quad \frac{\partial M}{\partial y}=0\\ \frac{\partial N}{\partial x}&=0, \quad \frac{\partial N}{\partial y}=3\end{aligned}\right\}\ \text{continuous everywhere}$$

And $\dfrac{\partial M}{\partial y}=0=\dfrac{\partial N}{\partial x}$.

Thus the equation is exact.

We need to find $f(x,y)$ where

$$\frac{\partial f}{\partial x}=M(x,y) \quad \text{and} \quad \frac{\partial f}{\partial y}=N(x,y)$$

That is we want to solve

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x}&=2x-1 \quad ①\\ \frac{\partial f}{\partial y}&=3y+7 \quad ②\end{aligned}}$$

Integrate (1) with respect to x to get

$$f(x,y) = x^2 - x + \underbrace{C(y)}_{\text{constant with respect to } x} \quad (3)$$

Integrate (2) with respect to y to get

$$f(x,y) = \frac{3}{2}y^2 + 7y + \underbrace{D(x)}_{\text{constant with respect to } y} \quad (4)$$

Set (3) equal to (4) to get

$$\underbrace{x^2 - x}\, + \underbrace{C(y)} = f(x,y) = \underbrace{\frac{3}{2}y^2 + 7y} + \underbrace{D(x)}$$

Set $D(x) = x^2 - x$ and $C(y) = \frac{3}{2}y^2 + 7y$.
Plug either into (3) or (4) to get f.
Plug into (3) to get

$$f(x,y) = x^2 - x + C(y) = x^2 - x + \frac{3}{2}y^2 + 7y$$

So a solution to the ODE is given implicitly by the equation

$$x^2 - x + \frac{3y^2}{2} + 7y = c$$

where c is a constant.

$\boxed{①(b)}$

$$\underbrace{5x+4y}_{M} + \underbrace{(4x-8y^3)}_{N}\, y' = 0$$

Let

$$\left.\begin{aligned} M(x,y) &= 5x+4y \\ N(x,y) &= 4x-8y^3 \end{aligned}\right\} \text{continuous everywhere}$$

Then,

$$\left.\begin{aligned} \frac{\partial M}{\partial x} &= 5, \quad \frac{\partial M}{\partial y} = 4 \\ \frac{\partial N}{\partial x} &= 4, \quad \frac{\partial N}{\partial y} = -24y^2 \end{aligned}\right\} \text{continuous everywhere}$$

We have that

$$\frac{\partial M}{\partial y} = 4 = \frac{\partial N}{\partial x}$$

So, the ODE is exact.

We want to find $f(x,y)$ where

$$\frac{\partial f}{\partial x} = M(x,y)$$

$$\frac{\partial f}{\partial y} = N(x,y)$$

So we need to solve

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= 5x+4y \quad (1) \\ \frac{\partial f}{\partial y} &= 4x-8y^3 \quad (2) \end{aligned}}$$

Integrate (1) with respect to x to get:

$$f(x,y) = \frac{5}{2}x^2 + 4yx + \underbrace{C(y)}_{\text{constant with respect to } x} \quad (3)$$

Integrate (2) with respect to y to get:

$$f(x,y) = 4xy - 2y^4 + \underbrace{D(x)}_{\text{constant with respect to } y} \quad (4)$$

Set (3) equal to (4) to get

$$\frac{5}{2}x^2 + \cancel{4yx} + C(y) = \cancel{4xy} - 2y^4 + D(x)$$

So,

$$\frac{5}{2}x^2 + C(y) = 2y^4 + D(x)$$

Set $C(y) = -2y^4$ and $D(x) = \frac{5}{2}x^2$

You can plug either into (3) or (4) to find f.

If you plug $C(y)$ into ③ you get:

$$f(x,y) = \frac{5}{2}x^2 + 4yx + C(y)$$

$$= \frac{5}{2}x^2 + 4yx - 2y^4$$

So an implicit solution to the ODE is given by the equation

$$\boxed{\frac{5}{2}x^2 + 4yx - 2y^4 = c}$$

Where c is any constant.

$\boxed{\text{(1)(c)}}$

$$\underbrace{-(x+6y)}_{N}y' + \underbrace{(2x+y)}_{M} = 0$$

Let
$$\left.\begin{aligned} M(x,y) &= 2x+y \\ N(x,y) &= -x-6y \end{aligned}\right\}\ \text{continuous everywhere}$$

Then,
$$\left.\begin{aligned} \frac{\partial M}{\partial x} = 2, \quad \frac{\partial M}{\partial y} = 1 \\ \frac{\partial N}{\partial x} = -1, \quad \frac{\partial N}{\partial y} = -6 \end{aligned}\right\}\ \text{continuous everywhere}$$

We have that
$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 1 \\ \frac{\partial N}{\partial x} &= -1 \end{aligned}\right\}\ \text{not equal}$$

Thus, the equation is <u>not</u> exact.

$\boxed{①(d)}$

$$\underbrace{\frac{2x}{y}}_{M} - \underbrace{\frac{x^2}{y^2}}_{N} \cdot \frac{dy}{dx} = 0$$

Let $M(x,y) = \dfrac{2x}{y} = 2xy^{-1}$

$N(x,y) = -\dfrac{x^2}{y^2} = -x^2y^{-2}$

continuous except when $y=0$

Then

$\dfrac{\partial M}{\partial x} = 2y^{-1}$, $\dfrac{\partial M}{\partial y} = -2xy^{-2}$

$\dfrac{\partial N}{\partial x} = -2xy^{-2}$, $\dfrac{\partial N}{\partial y} = 2x^2y^{-3}$

continuous except when $y=0$

Note that

$$\frac{\partial M}{\partial y} = -2xy^{-2} = \frac{\partial N}{\partial x}$$

equal when $y \neq 0$

Thus the equation is exact.

The solution will end up existing where $y \neq 0$ because of the above continuity notes.

We want to find f where

$$\frac{\partial f}{\partial x} = M(x,y)$$

$$\frac{\partial f}{\partial y} = N(x,y)$$

So we need to solve

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= 2xy^{-1} \quad (1) \\ \frac{\partial f}{\partial y} &= -x^2y^{-2} \quad (2) \end{aligned}}$$

Integrate (1) with respect to x to get:

$$f(x,y) = x^2y^{-1} + \underbrace{C(y)}_{\text{constant with respect to } x} \quad (3)$$

Integrate (2) with respect to y to get:

$$f(x,y) = x^2y^{-1} + \underbrace{D(x)}_{\text{constant with respect to } y} \quad (4)$$

Set (3) equal to (4) to get

$$x^2y^{-1} + C(y) = x^2y^{-1} + D(x)$$

So,

$$C(y) = D(x)$$

This implies $C(y)$ and $D(x)$ are both constants. You can make them both zero.

So, set $C(y) = 0$ and $D(x) = 0$.

Plug $C(y)$ into ③ to get

$$f(x,y) = x^2y^{-1} + C(y) = x^2y^{-1}$$

So, a solution to the ODE is given by

$$x^2y^{-1} = c \quad \text{or} \quad \frac{x^2}{y} = c$$

where c is any constant.

$\boxed{1(e)}$ $\underbrace{(2y^2x-3)}_{M}+\underbrace{(2yx^2+4)}_{N}y'=0$

Let

$$\left.\begin{aligned} M(x,y) &= 2y^2x-3 \\ N(x,y) &= 2yx^2+4 \end{aligned}\right\} \text{continuous everywhere}$$

Then

$$\left.\begin{aligned} \frac{\partial M}{\partial x} &= 2y^2 \quad & \frac{\partial M}{\partial y} &= 4yx \\ \frac{\partial N}{\partial x} &= 4yx \quad & \frac{\partial N}{\partial y} &= 2x^2 \end{aligned}\right\} \text{continuous everywhere}$$

And,

$$\frac{\partial M}{\partial y} = 4yx = \frac{\partial N}{\partial x}$$

So, the ODE is exact.

We must find f where

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= 2y^2x-3 \quad (1) \\ \frac{\partial f}{\partial y} &= 2yx^2+4 \quad (2) \end{aligned}}$$

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= M \\ \frac{\partial f}{\partial y} &= N \end{aligned}}$$

Integrate ① with respect to x to get

$$f(x,y) = y^2x^2 - 3x + C(y) \quad ③$$

constant with respect to x

Integrate ② with respect to y to get:

$$f(x,y) = y^2x^2 + 4y + D(x) \quad ④$$

constant with respect to y

Set ③ equal to ④ to get:

$$\cancel{y^2x^2} - 3x + C(y) = \cancel{y^2x^2} + 4y + D(x).$$

Thus,

$$-3x + C(y) = 4y + D(x)$$

Set $C(y) = 4y$ and $D(x) = -3x$.

Plug either into ③ or ④ to find f.

Plugging $C(y)$ into ③ we get

$$f(x,y) = y^2x^2 - 3x + C(y)$$
$$= y^2x^2 - 3x + 4y$$

So an implicit solution to the ODE is given by

$$y^2x^2 - 3x + 4y = c$$

where c is any constant.

$\boxed{1 (f)}$ Consider

$$\underbrace{\left(2y - \frac{1}{x} + \cos(3x)\right)}_{N} \frac{dy}{dx} + \underbrace{\frac{y}{x^2} - 4x^3 + 3y\sin(3x)}_{M} = 0$$

Let

$$\left.\begin{aligned} M(x,y) &= yx^{-2} - 4x^3 + 3y\sin(3x) \\ N(x,y) &= 2y - x^{-1} + \cos(3x) \end{aligned}\right\} \text{continuous everywhere except when } x = 0$$

Then,

$$\left.\begin{aligned} \frac{\partial M}{\partial x} &= -2yx^{-3} - 12x^2 + 9y\cos(3x) \\ \frac{\partial M}{\partial y} &= x^{-2} + 3\sin(3x) \\ \frac{\partial N}{\partial x} &= x^{-2} - 3\sin(3x) \end{aligned}\right\} \text{continuous everywhere except when } x = 0$$

$$\frac{\partial N}{\partial y} = 2$$

Note that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ except at discrete points when $\sin(3x) = 0$.

Thus the equation is <u>not</u> exact.

2 (a) From problem 1 above we saw that a solution to

$$(2x-1)+(3y+7)\frac{dy}{dx}=0$$

is given by the equation

$$x^2-x+\frac{3y^2}{2}+7y=c$$

We want the solution to satisfy $y(1)=2$.

plug $x=1, y=2$ into

$$x^2-x+\frac{3y^2}{2}+7y=c$$

to get

$$1^2-1+\frac{3(2)^2}{2}+7(2)=c$$

So,

$$20=c$$

Thus, a solution to the initial value problem is given by

$$\boxed{x^2-x+\frac{3}{2}y^2+7y=20.}$$

(2)(b) We are given that the equation

$$\underbrace{(e^x+y)}_{M} + \underbrace{(2+x+ye^y)}_{N} y' = 0$$

is exact.

Check:

$\frac{\partial M}{\partial y} = 1$

$\frac{\partial N}{\partial x} = 1$

equal

Let's find f where

$$\boxed{\begin{aligned} \frac{\partial f}{\partial x} &= e^x + y \quad (1) \\ \frac{\partial f}{\partial y} &= 2+x+ye^y \quad (2) \end{aligned}}$$

$\frac{\partial f}{\partial x} = M$

$\frac{\partial f}{\partial y} = N$

Integrate (1) with respect to x to get

$$f(x,y) = e^x + yx + \underbrace{C(y)}_{\text{constant with respect to } x} \quad (3)$$

Integrate (2) with respect to y to get

$$f(x,y) = 2y + xy + \int ye^y\,dy + \underbrace{D(x)}_{\text{constant with respect to } x}$$

We need $\int ye^y dy$. We get

$$\int ye^y dy = ye^y - \int e^y dy = ye^y - e^y$$

$u = y \quad du = dy$

$dv = e^y dy \quad v = e^y$

$\boxed{\int u\,dv = uv - \int v\,du}$

Thus,

$$f(x,y) = 2y + xy + ye^y - e^y + D(x) \quad (4)$$

Set (3) equal to (4) to get

$$e^x + \cancel{yx} + C(y) = \cancel{xy} + 2y + ye^y - e^y + D(x)$$

So,

$$\underbrace{e^x} + \underbrace{C(y)} = \underbrace{2y + ye^y - e^y} + \underbrace{D(x)}$$

Set $C(y) = 2y + ye^y - e^y$, $D(x) = e^x$

plug $C(y)$ into (3) to get

Thus,

$$f(x,y) = e^x + yx + C(y)$$
$$= e^x + yx + 2y + ye^y - e^y$$

So, an implicit solution to the ODE is given by the equation

$$e^x + yx + 2y + ye^y - e^y = c$$

where c is any constant.

We want the solution when $y(0) = 1$.

Plug in $x=0$, $y=1$ into the above to get

$$\underbrace{e^0}_{1} + \underbrace{1 \cdot 0}_{0} + \underbrace{2 \cdot 1}_{2} + \underbrace{1 \cdot e^1}_{e} - e^1 = c$$

Thus,

$$c = 3$$

So a solution to the initial value problem is given by

$$\boxed{e^x + yx + 2y + ye^y - e^y = 3}$$

② (c) We are given that the equation

$$\underbrace{\left(\frac{3y^2-x^2}{y^5}\right)}_{N}\frac{dy}{dx}+\underbrace{\frac{x}{2y^4}}_{M}=0$$

is exact.

Check:

$M=\frac{1}{2}xy^{-4} \rightarrow \frac{\partial M}{\partial y}=-2xy^{-5}$

$N=3y^{-3}-x^2y^{-5} \rightarrow \frac{\partial N}{\partial x}=-2xy^{-5}$

equal

We want f where

$$\frac{\partial f}{\partial x}=\frac{1}{2}xy^{-4} \quad ①$$

$$\frac{\partial f}{\partial y}=3y^{-3}-x^2y^{-5} \quad ②$$

$\frac{\partial f}{\partial x}=M$

$\frac{\partial f}{\partial y}=N$

Integrate ① with respect to x to get:

$$f(x,y)=\frac{1}{4}x^2y^{-4}+C(y) \quad ③$$

constant with respect to x

Integrate ② with respect to y to get:

$$f(x,y)=-\frac{3}{2}y^{-2}+\frac{1}{4}x^2y^{-4}+D(x) \quad ④$$

constant with respect to y

Set 3 equal to 4 to get

$$\cancel{\frac{1}{4}x^2y^{-4}} + c(y) = -\frac{3}{2}y^{-2} + \cancel{\frac{1}{4}x^2y^{-4}} + D(x)$$

So,

$$c(y) = -\frac{3}{2}y^{-2} + \underbrace{D(x)}_{0}$$

Set $c(y) = -\frac{3}{2}y^{-2}$ and $D(x) = 0$.

You can plug either into 3 or 4 to get f.

Plug $C(y) = -\frac{3}{2}y^{-2}$ into 3 to get

$$\begin{aligned} F(x,y) &= \frac{1}{4}x^2y^{-4} + c(y) \\ &= \frac{1}{4}x^2y^{-4} - \frac{3}{2}y^{-2} \end{aligned}$$

So a solution to the ODE is given by

$$\frac{1}{4}x^2y^{-4} - \frac{3}{2}y^{-2} = c$$

where c is any constant.

We want the solution when $y(1) = 1$.

So, plug $x=1, y=1$ into the above equation to get

$$\frac{1}{4}(1)^2(1)^{-4} - \frac{3}{2}(1)^{-2} = C$$

Thus,

$$C = \frac{1}{4} - \frac{3}{2} = -\frac{5}{4}$$

So a solution to the initial value problem is given by

$$\frac{1}{4}x^2y^{-4} - \frac{3}{2}y^{-2} = -\frac{5}{4}$$

or

$$\frac{x^2}{4y^4} - \frac{3}{2y^2} = -\frac{5}{4}$$